A Generalization of the Tanks-in-Series Mixing Model

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As an empirical model for nonideal flow in continuous flow systems, the cascade of n well-mixed tanks has proved useful in numerous applications. This amounts to characterizing the system by the transfer function:

$$\left(1+\frac{Vs}{nF}\right)^{-n}$$

In normalized form this becomes,

$$f_n(s) = \left(1+\frac{s}{n}\right)^{-n} \quad (1)$$

where time is expressed in units of the mean residence time V/F.

The parameter n is chosen to match a particular response: with n equal to one the model describes a single well-stirred vessel, while if the total system volume is held constant and n allowed to approach infinity it reduces to plug flow; an intermediate value of n may be used for the system whose behavior falls between these ideals. Equation (1) can, however, be inverted without restricting n to integral values; it is merely necessary that n be positive (*1*). The result is

$$f_n(T) = \frac{n^n\, T^{n-1}\, e^{-nT}}{\Gamma(n)} \quad (2)$$

so that the conventional interpretation of Equation (1) in terms of an integral number of well-mixed tanks places an unnecessary restriction on its application.

The purpose of this communication is to demonstrate how the usefulness of the model is increased when n is allowed to assume nonintegral values. Since the tanks do not exist physically in the situations where the model is applied there is no logical objection to this procedure.

APPLICATIONS

The most obvious advantage of using nonintegral values of n arises when fitting experimental response that lie between those for adjacent n values of the tanks-in-series model which is particularly inflexible for small n. Figure 1 shows a typical curve obtained from a tracer impulse test on a stirred vessel (*2*): Equation (2) with n equal to 1.07 provides a good single parameter description of this response; the small departure from perfect mixing is quite indescribable when using integral values of n.

An interesting situation arises when n is between zero and unity. Equation (2) then becomes rather like a rectangular hyperbola; the response is infinite at zero time, decays rapidly at first and finally more slowly than an exponential decay. Physical responses of this type occur when a fraction of the inflow bypasses all or a part of the system; and models that incorporate this effect have been devised for particular situations. Marr and Johnson (*3*) for example, in a model for a propeller-agitated vessel, allow for a fraction of the feed stream to pass directly to the outlet, so that the model response contains an impulse at zero time; and the model proposed by Gibilaro, Kropholler, and Spikins (*2*) to account for the effect of impeller speed and inlet feed position on the response of a turbine agitated blender, allows for a fraction of the inflow to bypass most of the vessel under certain conditions of operation. Figure 2 shows one of their typical experimental curves obtained under these conditions and compares it with Equation (2), n equal to 0.8. Again it is possible to get a quite reasonable fit by using the proposed generalization. In terms of well-mixed tank concepts this situation could be regarded as a fractional stirred tank.

Determination of n from experimental responses

A simple and reliable method for determining the most suitable value of n is to compare the normalized response

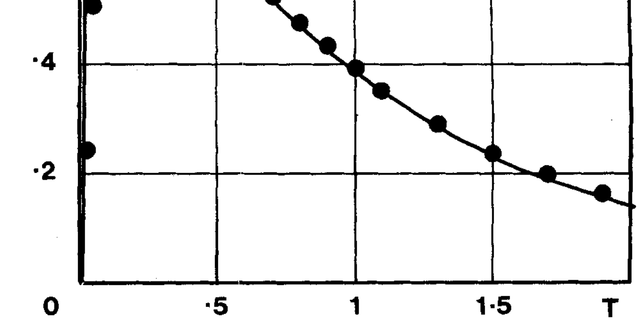

Fig. 1. Equation (2), $n = 1.07$, compared with experimental points presented in reference 2.

curve with curves calculated from Equation (2) for a range of values of n. If desired any appropriate goodness-of-fit criterion may be employed, but usually a subjective estimate is adequate. The following methods allow quick first estimates of n to be made.

Moments. The well-known relationship that the variance of the impulse-response of the tanks-in-series model is given by

$$\text{var}(T) = 1/n$$

applies also when n is nonintegral. The moments, however, are rather sensitive to experimental error so that the following methods may be more useful.

Peak time. Differentiation of Equation (2) leads to

$$n = 1/(1 - \mathbf{T}), \quad n \geq 1 \tag{3}$$

where **T** is the time at which the impulse response reaches its maximum value.

Response at a fixed normalized time. When $n < 1$ the previous is not applicable; an alternative is to use the response at some fixed normalized time. Figure 3 shows that $T = 0.5$ provides sufficient discrimination to be used to make an estimate of n for those cases where the initial response is very high.

DISCUSSION

The distribution $f_n(t)$ is known in the statistics literature as the gamma distribution. Koppel, et al. (*4*) have used the gamma distribution as the basis of a generalized penetration theory and present curves for several values of a parameter α which in our notation is equivalent to $n - 1$. As a mixing model, however, this generalization has apparently been overlooked and responses of the type shown in Figures 1 and 2 are generally described in terms of more complex models. Corrigan, et al. (*5*), for example, suggest adding another parameter to the tanks-in-series model to account for responses, like those of Figure 1, that fall between those of integral values of n; their modification consists of adding a recycle stream from the last tank of the tanks-in-series model back to the first tank so that by altering this recycle rate the response can be adjusted in a continuous manner. This model may be useful in situations where the recycle has some physical significance, and is very similar to a number of models proposed for nonideal stirred vessels, for instance that of Marr and Johnson (*3*) where the recycle is related to the pumping capacity of the impeller, but is quite unnecessary for interpolating between integral values of n in the tanks-in-series model. As demonstrated above, the transfer function of Equation (1) represents a continuous family of curves, and its restriction to integral n-values arises solely from the chemical engineer's preoccupation with stirred tanks.

The range of the model, as well as its sensitivity, is increased by allowing n to take any positive value: with n equal to zero the model indicates complete bypassing; the increase of n from zero represents successively less bypass flow until, with n equal to unity, perfect mixing is attained; the increase of n beyond unity yields a continuous family of curves representing successively less mixing and approaching plug flow as n approaches infinity.

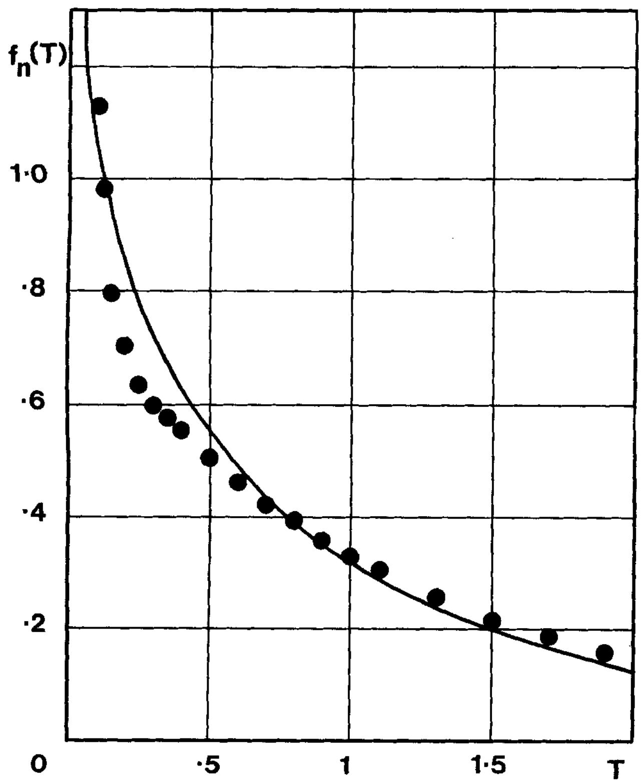

Fig. 2. Equation (2), $n = 0.8$, compared with experimental points presented in reference 2.

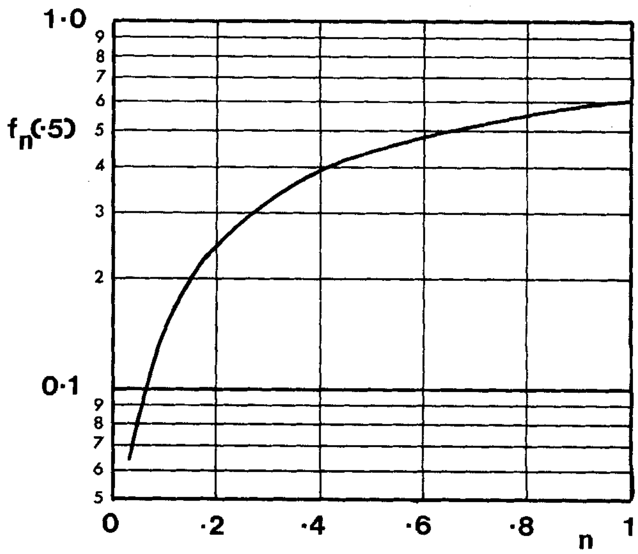

Fig. 3. Normalized response at $T = .5$, for $n < 1$.

NOTATION

F = throughput flow
$\mathbf{f}_n(s)$ = system transfer function
$f_n(T)$ = normalized impulse response
n = model parameter
s = Laplace variable
t = time
T = normalized time
$\mathbf{T}$ = normalized peak response time
V = system volume
var (T) = variance of $f_n(T)$
$\Gamma(n)$ = gamma function $\left(\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx\right)$

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